

# Chernoff's theorem for evolution families

Evelina Shamarova \*

February 3, 2008

## Abstract

A generalized version of Chernoff's theorem has been obtained. Namely, the version of Chernoff's theorem for semigroups obtained in a paper by Smolyanov, Weizsäcker, and Wittich [1] is generalized for a time-inhomogeneous case. The main theorem obtained in the current paper, Chernoff's theorem for evolution families, deals with a family of time-dependent generators of semigroups  $A_t$  on a Banach space, a two-parameter family of operators  $Q_{t,t+\Delta t}$  satisfying the relation:  $\frac{\partial}{\partial \Delta t} Q_{t,t+\Delta t} \Big|_{\Delta t=0} = A_t$ , whose products  $Q_{t_i,t_{i+1}} \dots Q_{t_{k-1},t_k}$  are uniformly bounded for all subpartitions  $s = t_0 < t_1 < \dots < t_n = t$ . The theorem states that  $Q_{t_0,t_1} \dots Q_{t_{n-1},t_n}$  converges to an evolution family  $U(s,t)$  solving a non-autonomous Cauchy problem. Furthermore, the theorem is formulated for a particular case when the generators  $A_t$  are time dependent second order differential operators. Finally, an example of application of this theorem to a construction of time-inhomogeneous diffusions on a compact Riemannian manifold is given.

*Keywords:* Chernoff's theorem, evolution family, strongly continuous semigroup, evolution families generated by manifold valued stochastic processes.

---

\*Institute for mathematical Methods in Economics, Vienna University of Technology

This work was supported by the research grant of the Erwin Schrödinger Institute for mathematical physics, and by the Austrian Science Fund (FWF) under START-prize-grant Y328.

Email: [Evelina.Shamarova@fam.tuwien.ac.at](mailto:Evelina.Shamarova@fam.tuwien.ac.at)

# 1 Chernoff's theorem for evolution families

## 1.1 Notation

Let  $A_t, t \in [S, T] \subset \mathbb{R}_+ \cup \{0\}$ , be generators of strongly continuous semigroups on a Banach space  $E$ . Let  $D(A_t)$  denote the domain of  $A_t$ . We assume that there exists a Banach space  $Y \subset \cap_{t \in [S, T]} D(A_t)$ , which is dense in  $E$ .

Given a  $t \in (S, T]$ , and an  $x \in Y$ , we consider a non-autonomous Cauchy problem on the interval  $[S, t]$  with the final condition  $x$ :

$$\begin{cases} \dot{u}(s) = -A_s u(s) \\ u(t) = x. \end{cases} \quad (1)$$

Let  $D_{[S, T]} = \{(s, t) : s \leq t, s \in [S, T], t \in [S, T]\}$ . The evolution family  $U(s, t), (s, t) \in D_{[S, T]}$ , solving the Cauchy problem (1) satisfies the relation:

$$U(s, r)U(r, t) = U(s, t) \quad (2)$$

for all  $s \leq r \leq t$  (see [5], Chapter VI, paragraph 2).

## 1.2 Case of non-commuting generators

**THEOREM 1** (Chernoff's theorem for evolution families). *Let  $A_t$  be generators of strongly continuous semigroups,  $Q_{t_1, t_2}, t_1, t_2 > 0$ , be a two-parameter family of bounded operators  $E \rightarrow E$ , and  $U(s, t), S \leq s \leq t \leq T$ , be an evolution family of operators with the generators  $A_t$  (see [5], Chapter VI, paragraph 2). We assume that the following assumptions are fulfilled:*

- 1) *there exists a Banach space  $Y \subset \cap_{t \in [S, T]} D(A_t)$  which is dense in  $E$ , invariant under the action of  $U(s, t)$  for all  $(s, t) \in D_{[S, T]}$ , i.e.  $U(s, t)Y \subset Y$ , and such that the Cauchy problem (1) is well-posed (backward solvable) for all  $x \in Y$ ;*
- 2) *the function  $[S, t] \rightarrow E, s \mapsto \frac{\partial}{\partial s} U(s, t)x$  is continuous for all  $x \in Y$  and  $t \in [S, T]$ ;*
- 3) *for any subinterval  $[s, t] \subset [S, T]$ , there exists a constant  $M(s, t) > 0$  such that for all sequences  $\{s = \tau_1 < \tau_2 < \dots < \tau_k \leq t\}$ ,  $\|Q_{\tau_1, \tau_2} \dots Q_{\tau_{k-1}, \tau_k}\| \leq M(s, t)$ ;<sup>1</sup>*

---

<sup>1</sup>For example, this assumption is fulfilled when  $Q_{\tau, \tau + \Delta\tau}$  are contractions.

4) for any subinterval  $[s, t] \in (S, T]$ , for any fixed  $x \in Y$ ,

$$\frac{Q_{\tau-\Delta\tau, \tau} - I}{\Delta\tau} U(\tau, t)x \rightarrow A_\tau U(\tau, t)x, \quad \Delta\tau \rightarrow 0 \quad (3)$$

uniformly in  $\tau \in [s, t]$ .

Then, for any subinterval  $[s, t] \subset [S, T]$ , for any sequence of partitions  $\{s = t_0 < t_1 < \dots < t_n = t\}$  of  $[s, t]$  such that  $\max(t_{j+1} - t_j) \rightarrow 0$  as  $n \rightarrow \infty$ , and for all  $x \in E$ ,

$$Q_{t_0, t_1} \dots Q_{t_{n-1}, t_n} x \rightarrow U(s, t)x, \quad n \rightarrow \infty.$$

*Proof.* First we consider the case  $s > S$ , i.e.  $[s, t] \subset (S, T]$ . We fix an arbitrary  $x \in Y$ . Using relation (2), we obtain:

$$\begin{aligned} & Q_{t_0, t_1} Q_{t_1, t_2} \dots Q_{t_{n-1}, t_n} - U(s, t) \\ &= \sum_{j=0}^{n-1} Q_{t_1, t_2} \dots Q_{t_{j-2}, t_{j-1}} (Q_{t_{j-1}, t_j} - U(t_{j-1}, t_j)) U(t_j, t). \end{aligned} \quad (4)$$

Let  $\delta_n = \max_j(t_j - t_{j-1})$  be the mesh of the partition  $\{s = t_0 < t_1 < \dots < t_n = t\}$ . Relation (4) implies the following inequality:

$$\begin{aligned} & \| (Q_{t_0, t_1} Q_{t_1, t_2} \dots Q_{t_{n-1}, t_n} - U(s, t))x \| \\ & \leq \sum_{j=0}^{n-1} \Delta t_j \left\| \left( \frac{Q_{t_{j-1}, t_j} - I}{t_j - t_{j-1}} - \frac{U(t_{j-1}, t_j) - I}{t_j - t_{j-1}} \right) U(t_j, t)x \right\|_E \\ & \leq (t - s) \sup \left\{ \left\| \left( \frac{Q_{\tau-\Delta\tau, \tau} - I}{\Delta\tau} - \frac{U(\tau-\Delta\tau, \tau) - I}{\Delta\tau} \right) U(\tau, t)x \right\|_E : \right. \\ & \quad \left. \tau \in [s, t], 0 < \Delta\tau < \delta_n \right\} \\ & \leq (t - s) \sup \left\{ \left\| \left( \frac{Q_{\tau-\Delta\tau, \tau} - I}{\Delta\tau} - A_\tau \right) U(\tau, t)x \right\|_E : \tau \in [s, t], 0 < \Delta\tau < \delta_n \right\} \end{aligned} \quad (5)$$

$$\begin{aligned} & + (t - s) \sup \left\{ \left\| \frac{U(\tau - \Delta\tau, t)x - U(\tau, t)x}{\Delta\tau} - A_\tau U(\tau, t)x \right\|_E : \right. \\ & \quad \left. \tau \in [s, t], 0 < \Delta\tau < \delta_n \right\} \end{aligned} \quad (6)$$

Assumption 4 implies that the term (5) converges to zero. Let us consider the term (6). Assumption 2 implies that the function

$$[s, t] \rightarrow E, \zeta \mapsto A_\zeta U(\zeta, t)$$

is continuous since  $\frac{\partial}{\partial \zeta} U(\zeta, t) = -A_\zeta U(\zeta, t)$ . Taking into consideration this, we obtain that for  $\Delta\tau \in (0, s - S)$  there exists a  $\theta \in (0, 1)$  such that

$$U(\tau - \Delta\tau, t)x = U(\tau, t)x + \Delta\tau A_{\tau - \theta\Delta\tau} U(\tau - \theta\Delta\tau, t)x. \quad (7)$$

Hence,

$$\begin{aligned} & \frac{U(\tau - \Delta\tau, t)x - U(\tau, t)x}{\Delta\tau} - A_\tau U(\tau, t)x \\ &= A_{\tau - \theta\Delta\tau} U(\tau - \theta\Delta\tau, t)x - A_\tau U(\tau, t)x \rightarrow 0, \quad \Delta\tau \rightarrow 0, \end{aligned}$$

where the right hand side converges to zero uniformly in  $\tau \in [s, t]$  since the continuous function  $[s, t] \rightarrow E, \zeta \mapsto A_\zeta U(\zeta, t)$  is uniformly continuous.

Thus, we have proved that  $Q_{t_0, t_1} \dots Q_{t_{n-1}, t_n} x \rightarrow U(s, t)x$  as  $n \rightarrow \infty$  for each  $x \in Y$  where  $Y$  is dense in  $E$ . Note that by Assumption 3 of the theorem, the bounded on  $E$  operators  $Q_{t_0, t_1} \dots Q_{t_{n-1}, t_n}$  are bounded uniformly in  $\{t_0, t_1, \dots, t_n\}$ . Hence, for an arbitrary subinterval  $[s, t] \subset (S, T]$ , the convergence  $Q_{t_0, t_1} \dots Q_{t_{n-1}, t_n} x \rightarrow U(s, t)x$  holds for all  $x \in E$ . Thus, we have proved the theorem for the case  $s > S$ .

Now we consider the case  $s = S$ . Let  $s_N$  be a decreasing system of real numbers such that  $\lim_{N \rightarrow \infty} s_N = s$ . For each fixed  $N$  and for all  $x \in E$ , we have:

$$Q_{t_0^N, t_1} \dots Q_{t_{n-1}, t_n} x \rightarrow U(s_N, t)x \quad \text{as } n \rightarrow \infty \quad (8)$$

where  $s_N = t_0^N < t_1 < \dots < t_n = t$  is a partition of  $[s_N, t]$ . Note that for each fixed  $\tau$  and for each fixed  $x$ ,  $\frac{1}{\Delta\tau} \|(Q_{\tau - \Delta\tau, \tau} - I)x\|_E$  is bounded, which follows from convergence (3) in Assumption 4 if we set  $t = \tau$ . By the Banach-Steinhaus theorem there exists a constant  $M_\tau > 0$  such that  $\frac{1}{\Delta\tau} \|Q_{\tau - \Delta\tau, \tau} - I\|_{E \rightarrow E} < M_\tau$ . This implies that  $\|Q_{\tau - \Delta\tau, \tau} - I\|_{E \rightarrow E}$  tends to zero as  $\Delta\tau \rightarrow 0$ . Let us fix an arbitrary  $\varepsilon > 0$ , and find a  $\delta > 0$  such that  $\|Q_{s, s_N} - I\|_{E \rightarrow E} < \varepsilon$  and  $\|U(s, s_N) - I\|_{E \rightarrow E} < \varepsilon$  whenever  $s_N - s < \delta$ . By Assumption 3 of Theorem 1,

$$\begin{aligned} & \|Q_{s, s_N} Q_{s_N, t_1} \dots Q_{t_{n-1}, t_n} - Q_{s_N, t_1} \dots Q_{t_{n-1}, t_n}\|_{E \rightarrow E} \\ & \leq \|Q_{s, s_N} - I\|_{E \rightarrow E} \|Q_{s_N, t_1} \dots Q_{t_{n-1}, t_n}\|_{E \rightarrow E} \leq M(s, t) \varepsilon. \end{aligned}$$

By continuity of  $U(\cdot, t)$ ,

$$\begin{aligned} \|U(s, t) - U(s_N, t)\|_{E \rightarrow E} &\leq \|U(s, s_N) - I\|_{E \rightarrow E} \|U(s_N, t)\|_{E \rightarrow E} \\ &\leq \sup_{\xi \in [S, t]} \|U(\xi, t)\|_{E \rightarrow E} \varepsilon \end{aligned}$$

Convergence (8) and two last estimates imply that for all  $x \in E$

$$Q_{t_0, t_1} \dots Q_{t_{n-1}, t_n} x \rightarrow U(s, t) x \quad \text{as } n \rightarrow \infty.$$

The theorem is proved.  $\square$

LEMMA 1. *Let  $Y \subset \cap_{\tau \in [S, T]} D(A_\tau)$  be a Banach space, dense in  $E$ , let  $[s, t] \subset (S, T]$  be a closed interval. Further let  $B_{\tau, \Delta\tau} : Y \rightarrow E$ ,  $\tau \in [s, t]$ ,  $\Delta\tau \in (0, s - S)$ , be bounded operators, and let  $U(\xi, \tau)$ ,  $(\xi, \tau) \in D_{[S, T]}$ , be an evolution family of operators with the generators  $A_\tau$ . We assume that the following assumptions are fulfilled:*

- 1) *for every  $y \in Y$ ,  $\|B_{\tau, \Delta\tau} y\|_E$  is bounded uniformly in  $\tau \in [s, t]$  and  $\Delta\tau \in [\delta, s - S]$ , where  $\delta \in (0, s - S)$  is fixed arbitrary;*
- 2)  *$U(\tau, t)Y \subset Y$  for all  $\tau \in [s, t]$ ;*
- 3) *for each  $y \in Y$ , the mapping  $[s, t] \rightarrow Y$ ,  $\tau \mapsto U(\tau, t)y$ , is continuous;*
- 4) *for each fixed  $y \in Y$ ,*

$$\sup_{\tau \in [s, t]} \|A_\tau y\|_E < \infty;$$

- 5) *for each fixed  $y \in Y$ ,*

$$\lim_{\Delta\tau \rightarrow 0} B_{\tau, \Delta\tau} y = A_\tau y \tag{9}$$

*where the convergence is uniform in  $\tau \in [s, t]$ .*

Then, for every  $y \in Y$ ,

$$\lim_{\Delta\tau \rightarrow 0} B_{\tau, \Delta\tau} U(\tau, t)y = A_\tau U(\tau, t)y \tag{10}$$

*and the convergence is uniform in  $\tau \in [s, t]$ .*

*Proof.* Assumptions 4 and 5, along with Assumption 1, imply that for each fixed  $y \in Y$ ,  $\|B_{\tau, \Delta\tau}y\|_E$  is bounded uniformly in  $\tau \in [s, t]$ , and  $\Delta\tau \in (0, s - S]$ . By the Banach-Steinhaus theorem,  $\|B_{\tau, \Delta\tau}\|_{Y \rightarrow E}$  are bounded uniformly in  $\tau \in [s, t]$  and  $\Delta\tau \in (0, s - S]$ , i.e. there exists a constant  $K$  such that

$$\|B_{\tau, \Delta\tau}\|_{Y \rightarrow E} < K.$$

We fix a  $y \in Y$ . The set

$$\{U(\tau, t)y, \tau \in [s, t]\} \quad (11)$$

is a compact in  $Y$  due to the continuity of the mapping  $[s, t] \rightarrow Y, \tau \mapsto U(\tau, t)y$ . Next, we fix an arbitrary small  $\varepsilon > 0$  and find a finite  $\varepsilon$ -net  $\{y_i\}_{i=1}^N \subset Y$  for the compact (11). Furthermore, we find a small  $\delta > 0$ , such that for all  $\tau \in [s, t]$ , for all  $\Delta\tau \in (0, \delta)$ , and for all  $y_i, 1 \leq i \leq N$ ,

$$\|B_{\tau, \Delta\tau}y_i - A_\tau y_i\|_E < \varepsilon.$$

Let  $\tau \in [s, t]$  be fixed arbitrary, and  $y_i$  be such that  $\|U(\tau, t)y - y_i\|_E < \varepsilon$ . We obtain:

$$\|B_{\tau, \Delta\tau}U(\tau, t)y - B_{\tau, \Delta\tau}y_i\|_E < K\|U(\tau, t)y - y_i\|_E < K\varepsilon.$$

Taking the limit in the right hand side, as  $\Delta\tau \rightarrow 0$ , we obtain

$$\|A_\tau U(\tau, t)y - A_\tau y_i\|_E \leq K\varepsilon.$$

This implies:

$$\begin{aligned} & \|B_{\tau, \Delta\tau}U(\tau, t)y - A_\tau U(\tau, t)y\|_E \\ & \leq \|B_{\tau, \Delta\tau}U(\tau, t)y - B_{\tau, \Delta\tau}y_i\|_E + \|B_{\tau, \Delta\tau}y_i - A_\tau y_i\|_E + \|A_\tau y_i - A_\tau U(\tau, t)y\|_E \\ & < (2K + 1)\varepsilon. \end{aligned}$$

This proves that the limit (10) exists and is uniform in  $\tau \in [s, t]$ . The lemma is proved.  $\square$

**THEOREM 2** (Corollary of Chernoff's Theorem). *Let  $A_t, Q_{t_1, t_2}, U(s, t)$ , and  $Y$  be as in Theorem 1. Let us assume that Assumptions 1–3 of Theorem 1 are fulfilled, and that for any subinterval  $[s, t] \subset (S, T]$ , Assumptions 3 and 4 of Lemma 1 are fulfilled. We assume, that for any  $y \in Y$ ,*

$$\lim_{\Delta\tau \rightarrow 0} \frac{Q_{\tau - \Delta\tau, \tau} - I}{\Delta\tau} y = A_\tau y$$

where the convergence is uniform in  $\tau$  running over closed subintervals  $[s, t] \subset (S, T]$ . Then, the statement of Theorem 1 holds true, i.e. for any subinterval  $[s, t] \subset [S, T]$ , for any sequence of partitions  $\{s = t_0 < t_1 < \dots < t_n = t\}$  of  $[s, t]$  such that  $\max(t_{j+1} - t_j) \rightarrow 0$  as  $n \rightarrow \infty$ , and for all  $x \in E$ ,

$$Q_{t_0, t_1} \dots Q_{t_{n-1}, t_n} x \rightarrow U(s, t) x, \quad n \rightarrow \infty.$$

*Proof.* Since we assume that Assumptions 1–3 of Theorem 1 are fulfilled, it suffices to prove that Assumption 4 of Theorem 1 is fulfilled. This will follow from Lemma 1 if we prove that Assumptions 1 and 5 of this lemma are fulfilled for the operators  $B_{\tau, \Delta\tau} = \frac{Q_{\tau-\Delta\tau, \tau} - I}{\Delta\tau}$ . Assumptions 2–4 of Lemma 1 clearly follow from those assumptions of Theorem 1 and Lemma 1 that are assumed here to be fulfilled. To prove Assumption 4 of Theorem 1, we fix an arbitrary closed interval  $[s, t] \subset (S, T]$ , and a  $\delta \in (0, s - S)$ . Then, for  $\Delta\tau \in [\delta, s - S]$ , we obtain:

$$\left\| \frac{Q_{\tau-\Delta\tau, \tau} - I}{\Delta\tau} \right\|_{E \rightarrow E} < \frac{M(s, t) + 1}{\delta}$$

where  $M(s, t)$  is the constant in Assumption 3 of Theorem 1. Assumption 5 of Lemma 1 is obviously fulfilled. By Lemma 1,

$$\lim_{\Delta\tau \rightarrow 0} \frac{Q_{\tau-\Delta\tau, \tau} - I}{\Delta\tau} U(\tau, t) y = A_\tau U(\tau, t) y$$

and the limit is uniform in  $\tau \in [s, t]$ . Applying Theorem 1 completes the proof of the theorem.  $\square$

### 1.3 Case of commuting generators

The following result has been obtained in [11] (p. 489, Proposition 2.5):

**PROPOSITION 1.** *Let  $\{A_t\}$  be a stable family of pairwise commuting generators of strongly continuous semigroups. Let us assume that there exists a space  $Y \subset \cap_{t \in [S, T]} D(A_t)$  which is dense in  $E$ , and let for all  $y \in Y$ , the mapping  $[S, T] \rightarrow E, t \mapsto A_t y$  be continuous. Then,  $(\int_s^t A_r dr, Y)$  is closable and its closure (which we still denote by  $\int_s^t A_r dr$ ) is a generator. Moreover, the Cauchy problem (1) is well-posed and the evolution family solving (1) is given by*

$$U(s, t) = e^{\int_s^t A_r dr}, \quad s \leq t.$$

**THEOREM 3** (Chernoff's theorem for evolution families). *Let  $A_t$  be a stable family of pairwise commuting generators of strongly continuous semigroups, and let  $Q_{t_1, t_2}$ ,  $t_1, t_2 > 0$ , be a two-parameter family of bounded operators  $E \rightarrow E$ , such that Assumptions 2–4 of Theorem 1 are fulfilled. Then, for any subinterval  $[s, t] \subset [S, T]$ , for any sequence of partitions  $\{s = t_0 < t_1 < \dots < t_n = t\}$  of  $[s, t]$  such that  $\max(t_{j+1} - t_j) \rightarrow 0$  as  $n \rightarrow \infty$ , and for all  $x \in E$ ,*

$$Q_{t_0, t_1} \dots Q_{t_{n-1}, t_n} x \rightarrow e^{\int_s^t A_r dr} x, \quad n \rightarrow \infty.$$

*Proof.* Proposition 1 implies that Cauchy problem (1) is well-posed, and that  $U(s, t) = e^{\int_s^t A_r dr}$  is the evolution family solving the Cauchy problem (1). Now the statement of the theorem follows immediately from Theorem 1.  $\square$

## 2 Chernoff's theorem for evolution families generated by manifold valued stochastic processes

Let  $M$  be a  $C^k$ -smooth compact manifold, and let  $A_0(t, x)$ ,  $A_1(t, x)$ ,  $\dots$ ,  $A_d(t, x)$ ,  $t \in [S, T]$ ,  $x \in M$ , be  $C^k$ -smooth vector fields on  $M$ . This means that if  $f \in C^j(M)$  and  $j > k$ , then  $A_i(t, \cdot)f \in C^k(M)$  and if  $j \leq k$ , then  $A_i(t, \cdot)f \in C^{j-1}(M)$  for all  $t \in [S, T]$ . Let us consider  $t$ -dependent second order differential operators:

$$A_t = \frac{1}{2} \sum_{\alpha=1}^d A_\alpha(t, \cdot) \circ A_\alpha(t, \cdot) + A_0(t, \cdot) \quad (12)$$

with the common domain  $C^k(M)$  independent of  $t$ . In the space  $C^k(M)$  we introduce the norm:

$$\|f\|_k = \sum_{|\lambda|=0}^k \sup_y \sup_{x \in V} |\partial^\lambda f \circ \psi_y(x)| \quad (13)$$

where  $\{(V, \psi_y), y \in M\}$  is an atlas covering  $M$ . The fact that  $\|\cdot\|_k$  defines a norm is proved in [9] (pp. 175-176). The space  $C^k(M)$  with the norm  $\|\cdot\|_k$  becomes a Banach space. We denote it by  $Y$ .



Given a probability space  $(\Omega, \mathcal{F}, \mathbb{P})$  with the filtration  $\mathcal{F}_t$ , and a  $d$ -dimensional  $\mathcal{F}_t$ -Brownian motion  $B_t^\alpha$ , we consider the stochastic differential equation:

$$\begin{cases} dX_t = A_\alpha(t, X_t) \circ dB_t^\alpha + A_0(t, X_t)dt \\ X_s = x \end{cases} \quad (14)$$

where  $A_\alpha(t, X_t) \circ dB_t^\alpha$  is the Stratonovich differential. We denote by  $\mathbb{E}$  the expectation relative to the measure  $\mathbb{P}$ . The operators  $A_t$  are generators of diffusions  $X_t$  on  $M$ .

LEMMA 2. *Let  $Y = (C^k(M), \|\cdot\|_k)$  where  $k \geq 3$ . Then, the solution of Cauchy problem (1) on the interval  $[S, t]$  with the generators (12) and with the final condition  $u(t, x) = f(x)$ ,  $f \in Y$ ,  $x \in M$ , exists, it is unique, and is given by*

$$u(s, x) = \mathbb{E}[f(X_t(s, x))]$$

where  $X_t(s, x)$  is the solution of SDE (14). Moreover,  $u(s, x) \in Y$ ,  $\frac{\partial}{\partial s}u(s, x) \in Y$  for all  $s \in [S, t]$ , and the mappings  $[S, t] \rightarrow Y, s \mapsto u(s, \cdot)$ , and  $[S, t] \rightarrow E, s \mapsto \frac{\partial}{\partial s}u(s, \cdot)$  are continuous.

*Proof.* Theorem 1.3 of Chapter 5 in [4] (p. 182) implies as a particular case that

$$u(s, x) = (U(s, t)f)(x) = \mathbb{E}[f(X_t(s, x))] \quad (15)$$

is a solution of the Cauchy problem (1). Here  $U(s, t)$ ,  $(s, t) \in D_{[S, T]}$ , is the evolution family solving this Cauchy problem. Consider another evolution family  $\tilde{U}(\tau, \xi)$ ,  $(\tau, \xi) \in D_{[S, T]}$ , satisfying the relation  $U(s, t) = \tilde{U}(T + S - t, T + S - s)$ . Evidently, there exists another SDE of type (14) with  $C_k$ -smooth coefficients, having a unique solution  $\tilde{X}_\xi(\tau, x)$ , such that for all  $f \in Y$ , for all  $x \in M$ ,

$$(\tilde{U}(\tau, \xi)f)(x) = \mathbb{E}[f(\tilde{X}_\xi(\tau, x))].$$

Applying Ito's formula gives:

$$\begin{aligned} (U(s, t)f)(x) &= (\tilde{U}(S + T - t, S + T - s)f)(x) = \mathbb{E}[f(\tilde{X}_{S+T-s}(S + T - t, x))] \\ &= f(x) + \int_0^{S+T-s} \mathbb{E}[(A_{S+T-\zeta}f)(\tilde{X}_\zeta(S + T - t, x))]d\zeta \end{aligned} \quad (16)$$

where we have exchanged the symbol  $\mathbb{E}$  for expectation with the integral in  $\zeta$  by Fubini's theorem. For the partial derivative in  $s$  we obtain:

$$\frac{\partial}{\partial s} u(s, x) = \frac{\partial}{\partial s} (U(s, t)f)(x) = -\mathbb{E}[(A_{S+T-s}f)(\tilde{X}_s(S+T-t, x))]. \quad (17)$$

Clearly,  $u(s, \cdot) \in Y$  and  $\frac{\partial}{\partial s} u(s, \cdot) \in Y$ . Also, relations (16) and (17) imply that the mappings  $[S, t] \rightarrow Y, s \mapsto u(s, \cdot)$  and  $[S, t] \rightarrow E, s \mapsto \frac{\partial}{\partial s} u(s, \cdot)$  are continuous. The lemma is proved.  $\square$

**THEOREM 4** (Chernoff's theorem for evolution families generated by manifold valued stochastic processes). *Let  $A_t, t \in [S, T]$ , be given by (12), and let  $D(A_t) = Y$  for all  $t$ . Further, let  $Q_{t_1, t_2}, S \leq t_1 < t_2 \leq T$ , be a family of contractions on  $C(M)$ . We assume that the following assumptions are fulfilled:*

- 1) *the functions  $[S, T] \rightarrow C(M), t \mapsto A_t f$  are continuously differentiable for all  $f \in Y$ ;*
- 2) *stochastic differential equation (14) has a unique solution  $X_t(s, x)$ ;<sup>2</sup>*
- 3) *for all  $f \in Y$ ,*

$$\lim_{\Delta\tau \rightarrow 0} \frac{Q_{\tau-\Delta\tau, \tau} - I}{\Delta\tau} f = A_\tau f$$

*and the limit is uniform in  $\tau$  running over closed intervals  $[s, t] \subset (S, T]$ .*

*Then, for any subinterval  $[s, t] \subset [S, T]$ , for any sequence of partitions  $\{s = t_0 < t_1 < \dots < t_n = t\}$  of  $[s, t]$  such that  $\max(t_{j+1} - t_j) \rightarrow 0$  as  $n \rightarrow \infty$ , and for all  $f \in C(M)$ , the following convergence holds in  $C(M)$ :*

$$(Q_{t_0, t_1} \dots Q_{t_{n-1}, t_n} f)(\cdot) \rightarrow \mathbb{E}[f(X_t(s, \cdot))], \quad n \rightarrow \infty.$$

*Proof.* Let  $[s, t] \subset (S, T]$  be fixed. We would like to apply Theorem 2. To this end, we have to verify Assumptions 1 – 3 of Theorem 1 and Assumptions 2 and 3 of Lemma 1. Assumption 1 of Theorem 1 follows from the paper [12] by Kato. The paper [12] guaranties existence and uniqueness of the solution of the Cauchy problem (1) if the following assumptions are fulfilled:

---

<sup>2</sup>Sufficient conditions under which (14) has a unique solution can be found for example in [4] and [13]

1)  $D(A_t) = Y$  for all  $t \in [S, T]$ , and  $Y$  is dense in  $E$ ; 2) the functions  $t \mapsto A_t f$  are continuously differentiable. Due to this result, Assumption 1 of Theorem 1 is fulfilled. Let  $U(s, t)$  be the evolution family solving the Cauchy problem (1), and let  $u(s, x)$  denote the solution of (1) with the final condition  $f(x)$  at time  $t$ . Assumption 2 of Theorem 1 is fulfilled by Lemma 2. Assumption 3 of Theorem 1 is fulfilled since  $Q_{t_1, t_2}$  are contractions. Assumptions 2 and 3 of Lemma 1 follow immediately from Lemma 2. Now the statement of the theorem is implied by Theorem 2.  $\square$

### 3 Example: a time-inhomogeneous manifold valued stochastic process constructed by Chernoff's theorem

Below, we describe a construction of a time-inhomogeneous Markov process on a compact Riemannian manifold using Theorem 4. Let  $M$  be a compact Riemannian manifold without boundary isometrically imbedded into  $\mathbb{R}^m$ , and  $\dim M = d$ . Let  $B_t$  be a Brownian motion on  $\mathbb{R}^m$  starting at the origin, and let  $\varphi : [0, 1] \rightarrow M$  be a two times continuously differentiable (non-random) function such that  $\varphi(0) = x$ . We consider the process  $W_t = B_t + \varphi(t)$ . Let  $\mathbb{W}_\varphi$  be its distribution,  $P_\varphi(t_1, z, t_2, A)$  be its transition probability. Clearly,

$$\begin{aligned} P_\varphi(t_1, z, t_2, A) &= P_0(t_1, z - \varphi(t_1), t_2, A - \varphi(t_2)) \\ &= \frac{1}{2\pi(t_2 - t_1)^{\frac{m}{2}}} \int_A e^{-\frac{|z - y - (\varphi(t_1) - \varphi(t_2))|^2}{2(t_2 - t_1)}} dy \end{aligned} \quad (18)$$

where  $P_0$  corresponds to the case when  $\varphi$  is equal to zero identically.

Let  $U_\varepsilon(M)$  be the  $\varepsilon$ -neighborhood of  $M$ , and let  $\mathbb{W}_{\varepsilon, t}^x$  be the distribution of the process which is conditioned to take a value in  $U_\varepsilon(M)$  at time  $t$ . Specifically, we define a measure  $\mathbb{W}_{\varepsilon, t}^x$  by the following expression on the right hand side:

$$\int_{C([0, t], \mathbb{R}^m)} f(\omega) \mathbb{W}_{\varepsilon, t}^x(d\omega) = \frac{\int_{C([0, t], \mathbb{R}^m)} \mathbb{I}_{\{\omega : \omega(t) \in U_\varepsilon(M)\}}(\omega) f(\omega) \mathbb{W}_\varphi^x(d\omega)}{\mathbb{W}_\varphi^x\{\omega : \omega(t) \in U_\varepsilon(M)\}}. \quad (19)$$

Let  $P_{\varepsilon, t}^x(\cdot, \cdot, \cdot, \cdot)$  be the transition probability for the distribution  $\mathbb{W}_{\varepsilon, t}^x$ . By

(18) and (19),  $P_{\varepsilon,t}^x(\cdot, \cdot, t, \cdot)$  is given by

$$\begin{aligned} \int_{\mathbb{R}^m} P_{\varepsilon,t}^x(s, z, t, dy) g(y) &= \int_{C([0,t], \mathbb{R}^m)} g(\omega(t)) \mathbb{W}_{\varepsilon,t}^x(d\omega) \\ &= \frac{\int_{U_\varepsilon(M)} e^{-\frac{|z-y-(\varphi(s)-\varphi(t))|^2}{2(t-s)}} g(y) dy}{\int_{U_\varepsilon(M)} e^{-\frac{|z-\bar{y}-(\varphi(s)-\varphi(t))|^2}{2(t-s)}} d\bar{y}} \end{aligned} \quad (20)$$

where  $g : \mathbb{R}^m \rightarrow \mathbb{R}$  is bounded and continuous. Obviously, as  $\varepsilon \rightarrow 0$ , the limit of the right hand side exists. Hence, the weak limit  $P_{[s,t]}$  of the measures  $P_{\varepsilon,t}^x(s, \cdot, t, \cdot)$  exists and equals

$$\begin{aligned} \int_{\mathbb{R}^m} P_{[s,t]}(z, dy) g(y) &= \frac{\int_M e^{-\frac{|z-y-(\varphi(s)-\varphi(t))|^2}{2(t-s)}} g(y) \lambda_M(dy)}{\int_M e^{-\frac{|z-\bar{y}-(\varphi(s)-\varphi(t))|^2}{2(t-s)}} \lambda_M(d\bar{y})} \\ &= \int_M q_\varphi(s, z, t, y) g(y) \lambda_M(dy) \end{aligned}$$

where  $\lambda_M$  is the volume measure on  $M$ , and

$$q_\varphi(s, z, t, y) = \frac{e^{-\frac{|z-y-(\varphi(s)-\varphi(t))|^2}{2(t-s)}}}{\int_M e^{-\frac{|z-\bar{y}-(\varphi(s)-\varphi(t))|^2}{2(t-s)}} \lambda_M(d\bar{y})}.$$

Given an interval  $[s, t]$ , the family of functions

$$p_\varphi(t_1, z, t_2, y) = \frac{1}{2\pi(t_2 - t_1)^{\frac{m}{2}}} e^{-\frac{|z-y-(\varphi(t_1)-\varphi(t_2))|^2}{2(t_2-t_1)}} \quad (21)$$

$s < t_1 < t_2 < t$ , together with the function  $q_\varphi(t_3, z, t, y)$ ,  $t_3 < t$ , builds a family of transition densities that defines the distribution of a Markov process on  $[s, t]$  conditioned to take a value on  $M$  at time  $t$ .

Consider a partition  $\mathcal{P} = \{s = t_0 < t_1 < \dots < t_n = t\}$ . For each partition interval  $[t_i, t_{i+1}]$ , for each pair of points  $\xi$  and  $\tau$  such that  $t_i < \xi < \tau \leq t_{i+1}$ , and for each Borel set  $A \subset \mathbb{R}^m$ , we define

$$Q(\xi, z, \tau, A) = \begin{cases} \int_A p_\varphi(\xi, z, \tau, y) dy, & \tau < t_{i+1}, \\ \int_{A \cap M} q_\varphi(\xi, z, \tau, y) \lambda_M(dy), & \tau = t_{i+1}. \end{cases} \quad (22)$$

Next, we add more points to the partition  $\mathcal{P}$  to obtain a partition  $\mathcal{P}' = \{s = \xi_0 < \xi_1 < \dots < \xi_N = t\}$  containing  $\mathcal{P}$ . The family of measures

$$Q(s, x, t, A) = \int_{\mathbb{R}^m} Q(s, x, \xi_1, dx_1) \int_{\mathbb{R}^m} Q(\xi_1, x_1, \xi_2, dx_2) \dots \int_{\mathbb{R}^m} Q(\xi_{N-2}, x_{N-2}, \xi_{N-1}, dx_{N-1}) Q(\xi_{N-1}, x_{N-1}, t, A)$$

is a family of transition probabilities for a Markov process starting at the point  $x \in M$  at time  $s$ , and conditioned to take values on  $M$  at all points of the partition  $\mathcal{P}$ .

We apply Theorem 4 to a subfamily of the family  $Q(\cdot, \cdot, \cdot, \cdot)$ . Specifically, we investigate weak convergence of the family

$$q_{\mathcal{P}}(s, x, t, y) = \int_M q_{\varphi}(s, x, t_1, x_1) \lambda_M(dx_1) \int_M q_{\varphi}(t_1, x_1, t_2, x_2) \lambda_M(dx_2) \dots \int_M q_{\varphi}(t_{n-2}, x_{n-2}, t_{n-1}, x_{n-1}) q_{\varphi}(t_{n-1}, x_{n-1}, t, y) \lambda_M(dx_{n-1}).$$

This family is a subfamily of  $Q(\cdot, \cdot, \cdot, \cdot)$  by definition (22) of the family  $Q$ . We consider the following two parameter family of contractions  $C(M) \rightarrow C(M)$ :

$$(Q_{t_i, t_{i+1}} f)(\cdot) = \int_M q_{\varphi}(t_i, \cdot, t_{i+1}, y) f(y) \lambda_M(dy). \quad (23)$$

**THEOREM 5.** *As the mesh of  $\mathcal{P}$  tends to zero, the following convergence holds in  $C(M)$ :*

$$\int_M q_{\mathcal{P}}(s, \cdot, t, y) g(y) \lambda_M(dy) \rightarrow \int_M p(s, \cdot, t, y) g(y) \lambda_M(dy) \quad (24)$$

where  $g \in C(M)$ ,  $p(s, x, t, y)$  is the transition density function of the process generated by

$$A_s = (\varphi'(s), \nabla_M)_{\mathbb{R}^m} - \frac{1}{2} \Delta_M. \quad (25)$$

**LEMMA 3.** *The  $A_s$  given by (25) generate contraction semigroups on  $C(M)$ . Moreover, each  $A_s$  is the generator of a diffusion  $X(\tau)$  on  $M$  which is the solution of the following SDE:*

$$\begin{cases} dr(\tau) = \tilde{L}_{\alpha}(r(\tau)) \circ dw_{s, \varphi}^{\alpha}, \\ r(0) = r, \end{cases} \quad (26)$$

where  $r(\tau) = (X^i(\tau), e_\alpha^i(\tau))$ ,  $\{e_\alpha(\tau)\}$  is a basis in the tangent space at the point  $X(\tau)$ ,  $\tilde{L}_\alpha$  are canonical horizontal vector fields [13],  $w_{s,\varphi}^\alpha(\tau) = \varphi'(s)^\alpha \tau + B^\alpha(\tau)$ ,  $B^\alpha(\tau)$  is a Brownian motion in  $\mathbb{R}^d$ .

*Proof.* Let  $r(\tau) = (X^i(\tau), e_\alpha^i(\tau))$  be the solution of (26). We find the generator of  $X(\tau)$ . Consider the function  $f(r) = f(x)$  for  $r = (x, e)$ . We have:

$$\begin{aligned} f(X(\tau)) - f(X(0)) &= f(r(\tau)) - f(r(0)) \\ &= \int_0^\tau (\tilde{L}_\alpha f)(r(\xi)) \circ dw_{s,\varphi}^\alpha(\xi) \\ &= \int_0^\tau \tilde{L}_\alpha f(r(\xi)) dB^\alpha(\xi) + \int_0^\tau \tilde{L}_\alpha f(r(\xi)) \varphi'(s)^\alpha d\xi \\ &\quad + \frac{1}{2} \int_0^\tau \sum_{\alpha=1}^d \tilde{L}_\alpha (\tilde{L}_\alpha f)(r(\xi)) d\xi. \end{aligned}$$

The definition of the generator of a process gives:

$$A_s f = \sum_{\alpha=1}^d (\tilde{L}_\alpha f, \varphi'(s)^\alpha)_{\mathbb{R}^d} + \frac{1}{2} \sum_{\alpha=1}^d \tilde{L}_\alpha (\tilde{L}_\alpha f).$$

Since  $f(r) = f(x)$ , i.e. does not depend on  $e$ , then the scalar product in the first term of the right hand side is well-defined, and

$$\tilde{L}f = \nabla_M f$$

by definition of  $\tilde{L}$ . Further, it was shown in [13] (Chapter V, paragraph 3) that

$$\sum_{\alpha=1}^d \tilde{L}_\alpha (\tilde{L}_\alpha f) = -\Delta_M f.$$

Thus, we have proved that

$$A_s f = (\varphi'(s), \nabla_M f)_{\mathbb{R}^d} - \frac{1}{2} \Delta_M f = (\varphi'(s), \nabla_M f)_{\mathbb{R}^m} - \frac{1}{2} \Delta_M f.$$

Since  $A_s$  is a generator of a diffusion on  $M$ ,  $A_s$  generates a contraction semigroup on  $C(M)$ . The lemma is proved.  $\square$

*Proof of Theorem 5.* We apply Theorem 4 to the generators (25) and the two-parameter family of contractions

$$(Q_{t_1, t_2} f)(x) = \int_M q_\varphi(t_1, x, t_2, y) f(y) \lambda_M(dy).$$

Assumption 1 of Theorem 4 is fulfilled by continuity of  $\varphi''(t)$  which we have assumed. Assumption 2 of Theorem 4 is fulfilled by Theorem 2.1, p. 152, from the book [4], where the authors have considered a more general case of a manifold in a Banach space. We show that Assumption 3 of Theorem 4 is fulfilled too. We have to prove that

$$\begin{aligned} \lim_{\Delta\tau \rightarrow 0} \frac{1}{\Delta\tau} \left( \int_M q_\varphi(\tau - \Delta\tau, y, \tau, z) g(z) \lambda_M(dz) - g(y) \right) \\ = (\varphi'(\tau), \nabla_M g(y))_{\mathbb{R}^m} - \frac{1}{2} \Delta_M g(y), \end{aligned} \quad (27)$$

and that the limit is uniform in  $\tau$ . Introduce the notation:  $\Delta\varphi_\tau = \varphi(\tau) - \varphi(\tau - \Delta\tau)$ .

LEMMA 4. *Let  $g \in C^3(M)$ . There exist a  $\delta > 0$ , a constant  $K_g > 0$ , and a function  $R : [0, \delta] \times M \times C^3(M) \rightarrow \mathbb{R}$  satisfying:*

$$|R(\Delta\tau, \cdot, g)| < K_g \Delta\tau^{\frac{1}{2}}, \quad (28)$$

and such that for all  $y \in M$  the following relation holds:

$$\begin{aligned} \frac{\int_M g(z) e^{-\frac{|z-y-\Delta\varphi_\tau|^2}{2\Delta\tau}} \lambda_M(dz)}{\int_M e^{-\frac{|z-y-\Delta\varphi_\tau|^2}{2\Delta\tau}} \lambda_M(dz)} = g(y) + (\Delta\varphi_\tau, \nabla_M g(y))_{\mathbb{R}^m} - \frac{\Delta\tau}{2} \Delta_M g(y) \\ + \Delta\tau R(\Delta\tau, y, g). \end{aligned} \quad (29)$$

*Proof.* We find a  $U_\varepsilon(M)$ , the  $\varepsilon$ -neighborhood of  $M$ , where the normal spaces  $N_{y_1}$  and  $N_{y_2}$  do not intersect each other for each pair of points  $y_1 \in M$  and  $y_2 \in M$ . Hence, each  $y \in U_\varepsilon(M)$  can be uniquely presented as  $y = z + tn(z)$ , where  $z \in M$ ,  $n(z) \in N_z$ , and  $|n(z)| = 1$ . Let  $P_M : U_\varepsilon(M) \rightarrow M$ ,  $z + tn(z) \mapsto z$ ,  $t \in \mathbb{R}$ , be the projection on  $M$ . For an arbitrary  $u \in \mathbb{R}^m$ ,  $|u| < \varepsilon$ , and a  $y \in M$ , we define:

$$u_M(y) = P_M(y + u) - y, \quad u_\perp(y) = u - u_M(y).$$

We have:

$$\frac{\int_M g(z) e^{-\frac{|z-y-\Delta\varphi_\tau|^2}{2\Delta\tau}} \lambda_M(dz)}{\int_M e^{-\frac{|z-y-\Delta\varphi_\tau|^2}{2\Delta\tau}} \lambda_M(dz)} = \frac{\int_M g(z) e^{-\frac{|z-y-(\Delta\varphi_\tau)_M(y)|^2}{2\Delta\tau}} e^{\frac{(z-y, (\Delta\varphi_\tau)_\perp(y))}{\Delta\tau}} \lambda_M(dz)}{\int_M e^{-\frac{|z-y-(\Delta\varphi_\tau)_M(y)|^2}{2\Delta\tau}} e^{\frac{(z-y, (\Delta\varphi_\tau)_\perp(y))}{\Delta\tau}} \lambda_M(dz)}. \quad (30)$$

We will need the following formula (see [1] and [2]):

$$\frac{\int_M e^{-\frac{|z-y|^2}{2t}} h(z) \lambda_M(dz)}{\int_M e^{-\frac{|z-y|^2}{2t}} \lambda_M(dz)} = h(y) - \frac{t}{2} \Delta_M h(y) + t \bar{R}(t, y, h) \quad (31)$$

where  $|\bar{R}(t, y, h)| < K \|h\|_3 t^{\frac{1}{2}}$ ,  $K$  is a constant, and the norms  $\|\cdot\|_3$  are defined by (13) for  $k = 3$ . Dividing the numerator and denominator in the right hand side of (30) by  $\int_M e^{-\frac{|z-y-(\Delta\varphi_\tau)_M(y)|^2}{2\Delta\tau}} \lambda_M(dz)$ , and applying (31), we obtain:

$$\begin{aligned} & \frac{\int_M g(z) e^{-\frac{|z-y-\Delta\varphi_\tau|^2}{2\Delta\tau}} \lambda_M(dz)}{\int_M e^{-\frac{|z-y-\Delta\varphi_\tau|^2}{2\Delta\tau}} \lambda_M(dz)} \\ &= \frac{g(y + (\Delta\varphi_\tau)_M(y)) - \frac{\Delta\tau}{2} \Delta_M \left( g e^{\frac{(\cdot - y, (\Delta\varphi_\tau)_\perp(y))}{\Delta\tau}} \right)(y) + \Delta\tau R_1}{1 - \frac{\Delta\tau}{2} \left( \Delta_M e^{\frac{(\cdot - y, (\Delta\varphi_\tau)_\perp(y))}{\Delta\tau}} \right)(y) + \Delta\tau R_2} \end{aligned} \quad (32)$$

where  $R_1$  and  $R_2$  are short-hand notations for  $\bar{R}(\Delta\tau, y + \Delta\varphi_\tau, g e^{\frac{(\cdot - y, (\Delta\varphi_\tau)_\perp(y))}{\Delta\tau}})$  and  $\bar{R}(\Delta\tau, y + \Delta\varphi_\tau, e^{\frac{(\cdot - y, (\Delta\varphi_\tau)_\perp(y))}{\Delta\tau}})$ , respectively. They can be estimated as follows:

$$|R_1| < \tilde{K} \|g\|_3 \|e^{\frac{(\cdot - y, (\Delta\varphi_\tau)_\perp(y))}{\Delta\tau}}\|_3 (\Delta\tau)^{\frac{1}{2}}, \quad |R_2| < K \|e^{\frac{(\cdot - y, (\Delta\varphi_\tau)_\perp(y))}{\Delta\tau}}\|_3 (\Delta\tau)^{\frac{1}{2}}.$$

We show that  $(\Delta_M e^{\frac{(\cdot - y, (\Delta\varphi_\tau)_\perp(y))}{\Delta\tau}})(y)$  and  $\|e^{\frac{(\cdot - y, (\Delta\varphi_\tau)_\perp(y))}{\Delta\tau}}\|_3$  are bounded in  $\tau$  and  $\Delta\tau$ . We consider local normal charts  $\psi(\bar{\xi}) = \psi(\xi_1, \dots, \xi_d)$  at the point  $y$ . We obtain:

$$\frac{\partial}{\partial \xi_i} e^{\frac{(\psi(\bar{\xi}) - y, (\Delta\varphi_\tau)_\perp(y))}{\Delta\tau}} = e^{\frac{(\psi(\bar{\xi}) - y, (\Delta\varphi_\tau)_\perp(y))}{\Delta\tau}} \left( \frac{\partial}{\partial \xi_i} \psi(\bar{\xi}), \frac{(\Delta\varphi_\tau)_\perp(y)}{\Delta\tau} \right)_{\mathbb{R}^m}. \quad (33)$$

This formula makes obvious the resulting expression upon taking two further derivatives, and thus, it shows that  $(\Delta_M e^{\frac{(\cdot - y, (\Delta\varphi_\tau)_\perp(y))}{\Delta\tau}})(y)$  and



$\|e^{\frac{(\cdot - y, (\Delta\varphi_\tau)_\perp(y))}{\Delta\tau}}\|_3$  are bounded in  $\tau$  and  $\Delta\tau$  if and only if  $\frac{(\Delta\varphi_\tau)_\perp(y)}{\Delta\tau}$  is bounded in  $\tau$  and  $\Delta\tau$ . The latter fact holds by existence of the limit:

$$\lim_{\Delta\tau \rightarrow 0} \frac{(\Delta\varphi_\tau)_\perp(y)}{\Delta\tau} = \text{Pr}_{N_y} \varphi'(\tau) \quad (34)$$

where  $\text{Pr}_{N_y}$  is the orthogonal projection onto  $N_y$ , the normal space at  $y$ . Now we can apply the short time asymptotic in  $\Delta\tau$  to the denominator at the right hand side of (32) while using the relation  $\Delta_M(g e^{\frac{(\cdot - y, (\Delta\varphi_\tau)_\perp(y))}{\Delta\tau}}) = e^{\frac{(\cdot - y, (\Delta\varphi_\tau)_\perp(y))}{\Delta\tau}} \Delta_M g - 2(\nabla_M e^{\frac{(\cdot - y, (\Delta\varphi_\tau)_\perp(y))}{\Delta\tau}}, \nabla_M g)_{\mathbb{R}^m} + g \Delta_M e^{\frac{(\cdot - y, (\Delta\varphi_\tau)_\perp(y))}{\Delta\tau}}$ . We obtain:

$$\begin{aligned} \frac{\int_M g(z) e^{-\frac{|z-y-\Delta\varphi_\tau|^2}{2\Delta\tau}} \lambda_M(dz)}{\int_M e^{-\frac{|z-y-\Delta\varphi_\tau|^2}{2\Delta\tau}} \lambda_M(dz)} &= g(y) + ((\Delta\varphi_\tau)_M(y), \nabla_M g(y))_{\mathbb{R}^m} - \frac{\Delta\tau}{2} \Delta_M g(y) \\ &\quad + \Delta\tau \left( \nabla_M e^{\frac{(\cdot - y, (\Delta\varphi_\tau)_\perp(y))}{\Delta\tau}} \Big|_y, \nabla_M g(y) \right)_{\mathbb{R}^m} + \Delta\tau \tilde{R}(\Delta\tau) \end{aligned} \quad (35)$$

where  $|\tilde{R}(\Delta\tau)| < \tilde{K} \|g\|_3 \Delta\tau^{\frac{1}{2}}$ ,  $\tilde{K}$  is a constant. Formulas (33) and (34) imply:

$$\left( \nabla_M e^{\frac{(\cdot - y, (\Delta\varphi_\tau)_\perp(y))}{\Delta\tau}} \Big|_y, \nabla_M g(y) \right) = \left( \frac{(\Delta\varphi_\tau)_\perp(y)}{\Delta\tau}, \nabla_M g(y) \right).$$

Substituting this in (35), we obtain (29). The lemma is proved.  $\square$

We continue the proof of Theorem 5. Lemma 4 easily implies the convergence in (27). This convergence is uniform in  $\tau$ . Indeed, (28) implies that  $R(\Delta\tau, \cdot, g)$  converges to zero uniformly in  $\tau$ . Further, we have:

$$\left| \frac{\Delta\varphi_\tau}{\Delta\tau} - \varphi'(\tau) \right| \leq \frac{1}{2} \sup_{\theta \in [0,1]} |\varphi''(\tau + \theta\Delta\tau)| \Delta\tau,$$

and hence,  $\frac{\Delta\varphi_\tau}{\Delta\tau}$  converges to  $\varphi'(\tau)$  uniformly in  $\tau$ . Thus, we have verified all the assumptions of Theorem 4. Finally, we note that since  $p(\cdot, \cdot, \cdot, \cdot)$  is the transition density for the diffusion process  $X_t(s, x)$ , and hence

$$\int_M p(s, x, t, y) f(y) \lambda_M(dy) = \mathbb{E}[f(X_t(s, x))]. \quad (36)$$

Now the statement of the theorem is implied by convergence (24), formulas (36) and (23), and by Theorem 4.  $\square$

## References

- [1] *Smolyanov O.G., Weizsäcker H.v., Wittich O.*, Brownian motion on a manifold as a limit of stepwise conditioned standard Brownian motions, Canadian Mathematical Society, Conference Proceedings, Vol. 29, 2000, pp. 589-602.
- [2] *É. Yu. Shamarova*, Constructing a Brownian sheet with values in a compact Riemannian manifold, (Russian, English), Math. Notes 76, No. 4, 590-596 (2004); translation from Mat. Zametki 76, No. 4, 635-640, 2004.
- [3] *Chernoff, R.*, Product formulas, Nonlinear semigroups, and Addition of Unbounded operators, Memoirs of American Math. Soc., 140, 1974
- [4] *Dalecky Yu. L, Belopolskaya Ya. I.*, Stochastic equations and differential geometry, Series: Mathematics and its Applications, Kluwer Academic Publishers, Netherlands, 260 p., 1989
- [5] *Dalecky Yu. L, Fomin S. V.*, Measures and differential equations in infinite-dimensional space, Series: Mathematics and its Applications, Kluwer Academic Publishers, Netherlands, 337 p., 1991
- [6] *Egorov Yu. V., Shubin M.A.*, Partial Differential Equations, Vol III, Springer-Verlag
- [7] *Engel, K.-J., Nagel R.*, One-parameter semigroups for linear evolution equations. (English) Graduate Texts in Mathematics. 194. Berlin: Springer. xxi, 586 p., 2000.
- [8] *Goldstein, Jerome A.*, Semigroups of linear operators and applications. (English) Oxford Mathematical Monographs. New York: Oxford University Press; Oxford: Clarendon Press. X, 245 p., 1985.
- [9] *Lawson, H. B., Michelsohn M.-L.*, Spin Geometry, Princeton University Press, Princeton NJ, 1989.
- [10] *Tanabe, Hiroki*, Equations of evolution. Translated from Japanese by N. Mugibayashi and H. Haneda. (English) Monographs and Studies in Mathematics. 6. London - San Francisco - Melbourne: Pitman. XII, 1979, 260 p.

- [11] *Nickel, Gregor; Schnaubelt, Roland*, An extension of Kato's stability condition for nonautonomous Cauchy problems. Taiwanese J. Math. 2, No.4, 483-496, 1998.
- [12] *Kato, Tosio*, Integration of the equation of evolution in a Banach space. (English) J. Math. Soc. Japan 5, 208-234 (1953).
- [13] *Ikeda N., Watanabe S.*, Stochastic differential equations and diffusion processes, North Holland publishing company (1989).